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A bosonization procedure for Hamiltonian theories with fermions

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Abstract. Hamiltonian theories with fermions are proved to be equivalent to hierarchies of ordinary Hamiltonian theories. The corresponding Poisson brackets are defined in terms of the original super-Poisson structure, while Hamiltonian functions are simply the coefficients in the expansion of the super-Hamiltonian function as a formal power series in Grassmann generators. Fermion extensions of the KdV equation are considered to illustrate the general result; its space-supersymmetric extensions are used to show in particular how supersymmetry transformations can be recast as ordinary Hamiltonian symmetries.

1. Introduction

The problem of defining a more or less formal classical limit for fermion systems has a long history [1]. Apart from its merely speculative meaning, it is relevant to the Feynman path integral quantization method, where the notion of classical phase, or configuration, space plays a crucial role [2]. As is well known, the path integral quantization procedure was extended long ago to fermion systems by Berezin in terms of a purely algebraic notion of the integral on fermion degrees of freedom [3]. This approach led to the introduction of the concept of a (pseudo-)classical superdynamical system [4], as a system with both traditional bosonic and anticommuting fermionic variables; this notion became widely popular among physicists in particular due to the emergence of superstring theory [5].

As to the generalization of classical dynamics mentioned above, several attitudes are possible. The simplest choice is to work in a purely algebraic setting in analogy to the Berezin approach to the Feynman path integral on fermion variables. In this context the formulation of classical dynamics in terms of derivations on the ring of smooth functions on a given phase manifold, or related settings, can be taken as the starting point [6]†. It is then quite natural to generalize this notion of the classical dynamical system, just taking more general non-Abelian rings as dynamical variable sets. In particular a Z_2 graded ring $\mathscr{F}$, i.e. such that

$$\varphi\psi=(-1)^{g(\varphi)g(\psi)}\psi\varphi \tag{1}$$

where φ, ψ are pure elements, i.e. of definite grading $g(\varphi), g(\psi)=0, 1\in Z_2$, is presumably the most general arena for superdynamics. However, this extremely general axiomatic

† A related aspect consists of the algebraic characterization of tangent and cotangent bundles and correspondingly of their Lagrangian and Hamiltonian vector fields [7].

context has several limitations, the most severe being the lack, without further specifications, of the notion of flow corresponding to a given supervector field (derivation). Furthermore it deprives path integral quantization of its strongly advocated probabilistic interpretation, which is already present in the original Feynman conception [8] and, strictly speaking, is rigorously established for its Euclidean version [9].

What can be considered in some sense the opposite constructive viewpoint, was recently applied to show how some graded extensions of the KdV equation can be recast in terms of ordinary Hamiltonian field theories, which in particular admit bi-Hamiltonian structure if the corresponding superdynamics does [10]. It was also shown that the ordinary Hamiltonian systems constructed starting from the space-supersymmetric versions of the KdV equation admit, in correspondence with the original supersymmetry transformations, ordinary Hamiltonian symmetries. The aim of the present paper is to prove that this construction works in general for arbitrary super-Hamiltonian systems, by giving the general prescription for obtaining the ordinary Hamiltonian structures from the super-Hamiltonian ones. The general results will then be applied to the $1+1$ dimensional superfield theories already mentioned.

The motivation is twofold: first, the proposed transcription of superdynamics, in terms of ordinary dynamics only containing commuting variables, removes the subtleties inherent in the very notions of time flow and super-Hamiltonian integrability; second, it leads to the possibility of defining the path integral on fermion variables in terms of ordinary measure-theoretic integration [11]. This last point is particularly relevant since it recovers the probabilistic interpretation and then, for instance, avoids the need for *ad hoc* tricks to perform Monte Carlo simulations of fermion systems [12].

To fix language and notation, consider the traditional context in which superdynamics is usually formulated. The implied ring $\mathscr{F}$ is generated by two families $(u_\alpha)_{\alpha\in B}, (\varphi_\beta)_{\beta\in F}$ of pure elements

$$\mathrm{g}(u_\alpha)=0,\ \alpha\in B;\ \mathrm{g}(\varphi_\beta)=1 \qquad \beta\in F \tag{2}$$

respectively corresponding to bosonic and fermionic degrees of freedom. The families B and F of bosonic and fermionic indices (which are in principle independent, except for supersymmetric theories), can be either finite or infinite; in particular they are formally considered non-denumerably infinite if field theories are involved. Time evolution is then defined by putting

$$\dot{u}_\alpha = X_\alpha(\boldsymbol{u},\boldsymbol{\varphi}) \qquad \alpha\in B \tag{3a}$$

$$\dot{\varphi}_\beta = Y_\beta(\boldsymbol{u},\boldsymbol{\varphi}) \qquad \beta\in F \tag{3b}$$

where X_α, Y_β are in general formal series in their arguments and mostly just polynomials, while here and henceforth bold characters $\boldsymbol{u}$ and $\boldsymbol{\varphi}$ denote the whole families of Bose and Fermi variables. The above equations are usually considered as transcriptions in local coordinates of a dynamical equation for a vector field globally defined (or in principle globally definable) in some suitable supermanifold [13]. Here only the local aspects will be considered; the global analysis of the present proposal will be treated elsewhere [14].

The present paper is concerned in particular with super-Hamiltonian dynamics; in such a case the RHSs of equations (3*a*) and (3*b*) are assumed to be local components of a super-Hamiltonian vector field corresponding to a specific super-Hamiltonian structure and super-Hamiltonian function H, which is a suitable even (i.e. $\mathrm{g}(H)=0$) element of $\mathscr{F}$. To be specific, a super-Hamiltonian structure on $\mathscr{F}$ is given in terms of a super-Poisson bracket, i.e. a graded antisymmetric c-map

$$(f, g) \in \mathscr{F} \times \mathscr{F} \mapsto \{f, g\} \in \mathscr{F} \tag{4}$$

satisfying the graded Jacobi identity and being a derivation with respect to both arguments. To be a graded antisymmetric c-map means that

$$\{g, f\} = -(-1)^{\mathfrak{g}(f)\mathfrak{g}(g)}\{f, g\} \tag{5}$$

$$\mathfrak{g}(\{f, g\}) = \mathfrak{g}(fg) = \mathfrak{g}(f) + \mathfrak{g}(g) \pmod 2 \tag{6}$$

while the graded Jacobi identity and derivation property respectively read

$$(-1)^{\mathfrak{g}(f)\mathfrak{g}(h)}\{f, \{g, h\}\} + (-1)^{\mathfrak{g}(h)\mathfrak{g}(g)}\{h, \{f, g\}\} + (-1)^{\mathfrak{g}(g)\mathfrak{g}(f)}\{g, \{h, f\}\} = 0 \tag{7a}$$

$$\{f, gh\} = (-1)^{\mathfrak{g}(f)\mathfrak{g}(g)}g\{f, h\} + \{f, g\}h \tag{7b}$$

where f, g, h are pure elements of $\mathscr{F}$. The super-Hamiltonian character of equations (3a) and (3b) then means in particular that their RHSs are given by

$$X_\alpha(\boldsymbol{u}, \boldsymbol{\varphi}) = \{u_\alpha, H\} \tag{8a}$$

$$Y_\beta(\boldsymbol{u}, \boldsymbol{\varphi}) = \{\varphi_\beta, H\} \tag{8b}$$

and in general that, for a generic element f of $\mathscr{F}$,

$$\dot{f} = \{f, H\}. \tag{8c}$$

2. From super-Hamiltonian back to Hamiltonian systems

In order to recast super-Hamiltonian dynamics in an ordinary Hamiltonian setting, consider explicitly the u and φ as local coordinates on a supermanifold locally modelled on $\mathbb{C}_c^{\#(B)} \times \mathbb{C}_a^{\#(F)}$, $\#(B)$ and $\#(F)$ respectively denoting the (possibly infinite) number of boson and fermion degrees of freedom [13]; here $\mathbb{C}_c$ denotes the subalgebra of commuting supernumbers, $\mathfrak{g}(\mathbb{C}_c) = 0$, and $\mathbb{C}_a$ the subspace of anticommuting ones, $\mathfrak{g}(\mathbb{C}_a) = 1$. This means that the u and φ can be represented as follows:

$$u_\alpha = u_{\alpha,B} + \sum_k \frac{1}{(2k)!} u_{\alpha,(j_1,j_2,\ldots,j_{2k})}\zeta_{j_1}\zeta_{j_2}\ldots\zeta_{j_{2k}} \tag{9a}$$

$$\varphi_\beta = \sum_k \frac{1}{(2k-1)!} \varphi_{\beta,(j_1,j_2,\ldots,j_{2k-1})}\zeta_{j_1}\zeta_{j_2}\ldots\zeta_{j_{2k-1}} \tag{9b}$$

where $u_{\alpha,B}$ is the body of u_α, while the sum in (9a) is its soul $u_{\alpha,S}$ and the summation is implied on repeated Grassmann indices. Here $\zeta_1, \zeta_2, \ldots, \zeta_k, \ldots$, with $\zeta_i\zeta_j + \zeta_j\zeta_i = 0$, are a family of generators of the Grassmann algebra Λ, whose dimensionality is irrelevant in what follows. If in particular $\Lambda = \Lambda_\infty$, which is always implicitly assumed in field theory in order for the corresponding quantum theory to be able to accommodate the whole Fock space, expressions given in equations (9a) and (9b) are to be meant as formal series, thus avoiding any notion of convergence in Λ_∞ [13]. Finally, power series coefficients in equations (9a) and (9b) are ordinary complex variables, completely antisymmetric in their Grassmann indices. Here for simplicity they are assumed to be real, which does not imply that u and φ are real supernumbers, since, although Grassmann generators are taken as usual to be real, i.e. $\zeta_k^* = \zeta_k$, their products $\zeta_1\zeta_2\ldots\zeta_k$ are real or imaginary according to the value of k [13].

As to functions X_α and Y_β appearing in equations (3a) and (3b), it is worth remembering that they are defined as power series in φ and in the souls of u [13]; once expressions given in equations (9a) and (9b) are substituted for u and φ in these series, X_α and Y_β are given as (formal) power series in Grassmann generators as follows:

$$X_\alpha(\boldsymbol{u},\varphi)=X_{\alpha,B}(\boldsymbol{u}_B)+\sum_k\frac{1}{(2k)!}X_{\alpha,(j_1,j_2,\ldots,j_{2k})}(\boldsymbol{u}_{[c]},\boldsymbol{\varphi}_{[d]})\zeta_{j_1}\zeta_{j_2}\ldots\zeta_{j_{2k}} \tag{10a}$$

$$Y_\beta(\boldsymbol{u},\boldsymbol{\varphi})=\sum_k\frac{1}{(2k-1)!}Y_{\beta,(j_1,j_2,\ldots,j_{2k-1})}(\boldsymbol{u}_{[c]},\boldsymbol{\varphi}_{[d]})\zeta_{j_1}\zeta_{j_2}\ldots\zeta_{j_{2k-1}} \tag{10b}$$

where power series coefficients are oridnary complex variables, fulfilling the same antisymmetry and reality (for consistency) requirements as for coefficients in equations (9a) and (9b), and $[c]$, $[d]$ denote generic (respectively even and odd) Grassmann multi-indices, i.e. ordered subsets of $\{j_1,j_2,\ldots,j_{2k}\}$ in equation (10a) and $\{j_1,j_2,\ldots,j_{2k-1}\}$ in equation (10b). Once expressions (9a), (9b), (10a) and (10b) are substituted for left- and right-hand sides in equations (3a) and (3b), an equivalent ordinary differential system for real coefficients of the expansions (9a) and (9b), from now on called component variables, is obtained:

$$\dot{u}_{\alpha,B}=X_{\alpha,B}(\boldsymbol{u}_B) \tag{11a}$$

$$\dot{\varphi}_{\alpha,(i)}=Y_{\alpha,(i)}(\boldsymbol{u}_B,\boldsymbol{\varphi}_{(i)}) \tag{11b}$$

$$\dot{u}_{\alpha,(i,j)}=X_{\alpha,(i,j)}(\boldsymbol{u}_B,\boldsymbol{u}_{(i,j)},\boldsymbol{\varphi}_{(i)},\boldsymbol{\varphi}_{(j)}) \tag{11c}$$

$$\dot{\varphi}_{\alpha(i,j,h)}=Y_{\alpha,(i,j,h)}(\boldsymbol{u}_B,\boldsymbol{u}_{(i,j)},\boldsymbol{u}_{(i,h)},\boldsymbol{u}_{(j,h)},\boldsymbol{\varphi}_{(i)},\boldsymbol{\varphi}_{(j)},\boldsymbol{\varphi}_{(h)},\boldsymbol{\varphi}_{(i,j,h)}) \tag{11d}$$

$$\begin{aligned}\dot{u}_{\alpha,(i,j,h,k)}=X_{\alpha,(i,j,h,k)}(&\boldsymbol{u}_B,\boldsymbol{u}_{(i,j)},\boldsymbol{u}_{(i,h)},\boldsymbol{u}_{(i,k)},\boldsymbol{u}_{(j,h)},\boldsymbol{u}_{(j,k)},\boldsymbol{u}_{(h,k)},\boldsymbol{u}_{(i,j,h,k)},\\&\boldsymbol{\varphi}_{(i)},\boldsymbol{\varphi}_{(j)},\boldsymbol{\varphi}_{(h)},\boldsymbol{\varphi}_{(k)},\boldsymbol{\varphi}_{(i,j,h)},\boldsymbol{\varphi}_{(i,j,k)},\boldsymbol{\varphi}_{(i,h,k)},\boldsymbol{\varphi}_{(j,h,k)})\end{aligned} \tag{11e}$$

$$\vdots$$

If equations (3a) and (3b) are assumed to define a super-Hamiltonian system according to equations (8a) and (8b), then coefficients in the expansion of the super-Hamiltonian function

$$\begin{aligned}H(\boldsymbol{u},\boldsymbol{\varphi})&=H_B(\boldsymbol{u}_B)+\sum_k\frac{1}{(2k!)}H_{(j_1,j_2,\ldots,j_{2k})}\zeta_{j_1}\zeta_{j_2}\ldots\zeta_{j_{2k}}\\&=H_B(\boldsymbol{u}_B)+\frac{1}{2!}H_{(i,j)}(\boldsymbol{u}_B,\boldsymbol{u}_{(i,j)},\boldsymbol{\varphi}_{(i)},\boldsymbol{\varphi}_{(j)})\zeta_i\zeta_j\\&\quad+\frac{1}{4!}H_{(i,j,h,k)}(\boldsymbol{u}_B,\boldsymbol{u}_{(i,j)},\boldsymbol{u}_{(i,h)},\boldsymbol{u}_{(i,k)},\boldsymbol{u}_{(j,h)},\boldsymbol{u}_{(j,k)},\boldsymbol{u}_{(h,k)},\boldsymbol{u}_{(i,j,h,k)},\\&\quad\boldsymbol{\varphi}_{(i)},\boldsymbol{\varphi}_{(j)},\boldsymbol{\varphi}_{(h)},\boldsymbol{\varphi}_{(k)},\boldsymbol{\varphi}_{(i,j,h)},\boldsymbol{\varphi}_{(i,j,k)},\boldsymbol{\varphi}_{(i,h,k)},\boldsymbol{\varphi}_{(j,h,k)})\zeta_i\zeta_j\zeta_h\zeta_k+\ldots\end{aligned} \tag{12}$$

are constants of motion for system (11).

The main purpose of this paper is to prove that in the above hypothesis suitable closed subsystems of equations (11) form hierarchies of ordinary Hamiltonian systems with respect to ordinary Hamiltonian structures defined in terms of the original super-Poisson brackets. The ordinary Hamiltonian functions are given correspondingly by complex coefficients in expansion (12), which, consistently with the assumed reality of component variables, are taken to be real. To this end consider a finite subset of

$2n$ Grassmann indices, which, owing to the completely equivalent role played by all of them, can be identified without loss of generality with $\{1,2,\ldots,2n\}$. It can be proved that the closed subsystem of equations (11) including the evolution equations for component variables whose Grassmann multi-indices only contain indices up to $2n$, is Hamiltonian, with $H_{(1,2,\ldots,2n)}$ as the Hamiltonian function, with respect to a suitable Hamiltonian structure on the Abelian ring generated by the fixed set of component variables. In order to prove that and to find the Hamiltonian structure mentioned, it is obviously enough to work with a generic (even) monomial Hamiltonian function

$$M(u,\varphi)=u_{\alpha_1}u_{\alpha_2}\ldots u_{\alpha_h}\varphi_{\beta_1}\varphi_{\beta_2}\ldots\varphi_{\beta_{2k}}. \tag{13}$$

Now, in order for the closed subsystem under consideration to be Hamiltonian with $M_{(1,2,\ldots,2n)}$ as the hamiltonian function, the following relations have to hold:

$$X_{\alpha,[c]}=\{u_{\alpha,[c]},M_{(1,2,\ldots,2n)}\} \tag{14a}$$

$$Y_{\beta,[d]}=\{\varphi_{\beta,[d]},M_{(1,2,\ldots,2n)}\} \tag{14b}$$

(where Grassmann multi-indices $[c]$, $[d]$ are repsectively generic even and odd ordered subsets of $\{1,2,\ldots,2n\}$), or more explicitly

$$X_{\alpha,B}=\{u_{\alpha,B},M_{(1,2,\ldots,2n)}\} \tag{15a}$$

$$Y_{\beta,(i)}=\{\varphi_{\beta,(i)},M_{(1,2,\ldots 2n)}\} \tag{15b}$$

$$X_{\alpha,(i,j)}=\{u_{\alpha,(i,j)},M_{(1,2,\ldots,2n)}\} \tag{15c}$$

$$\vdots$$

$$X_{\alpha,(1,2,\ldots,2n)}=\{u_{\alpha,(1,2,\ldots,2n)},M_{(1,2,\ldots,2n)}\}. \tag{15d}$$

Here and henceforth the same symbol is used to denote both the original super-Poisson and the corresponding ordinary Poisson brackets we are looking for; what is meant is clear from their arguments.

On the other hand, if $\{u_\alpha,H\}_{[c]}$, $\{\varphi_\beta,H\}_{[d]}$ denote the generic coefficients in the expansion of the RHSs of equations ($8a$) and ($8b$) respectively, as formal power series in Grassmann generators, compatibility of equations (8) with M substituted for H and equations (14) reads

$$\{u_{\alpha,[c]},M_{(1,2,\ldots,2n)}\}=\{u_\alpha,M\}_{[c]} \tag{16a}$$

$$\{\varphi_{\beta,[d]},M_{(1,2,\ldots,2n)}\}=\{\varphi_\beta,M\}_{[d]}. \tag{16b}$$

In order to deduce from equations (16) the Poisson brackets we are looking for, consider first of all the explicit expressions of their RHSs, which appear as coefficients of the power series expansions in Grassmann generators of $\{u_\alpha,M\}$ and $\{\varphi_\alpha,M\}$. To be specific, if the derivation property, equation ($7b$), of super-Poisson brackets is used, it turns out that

$$\begin{aligned}\{u_\alpha,M\}_{[c]}=\sum_{p\in P([c],h,2k)}(-1)^{\pi(p)}\Bigg[&\sum_j u_{\alpha_1,[a_1]}\ldots\{u_\alpha,u_{\alpha_j}\}_{[a_j]}\\ &\times\ldots u_{\alpha_h,[a_h]}\varphi_{\beta_1,[b_1]}\ldots\varphi_{\beta_{2k},[b_{2k}]}\\ &+\sum_j u_{\alpha_1,[a_1]}\ldots u_{\alpha_h,[a_h]}\varphi_{\beta_1,[b_1]}\ldots\{u_\alpha,\varphi_{\beta_j}\}_{[b_j]}\ldots\varphi_{\beta_{2k},[b_{2k}]}\Bigg]\end{aligned} \tag{17a}$$

$$\{\varphi_\alpha, M\}_{[d]} = \sum_{p\in P([d],h-1,2k+1)} (-1)^{\pi(p)} \sum_j u_{\alpha_1,[a_1]} \ldots \{\varphi_\alpha, u_{\alpha_j}\}_{[a_j]}$$
$$\times \ldots u_{\alpha_h,[a_h]} \varphi_{\beta_1,[b_1]} \ldots \varphi_{\beta_{2k},[b_{2k}]}$$
$$+ \sum_{p\in P([d],h+1,2k-1)} (-1)^{\pi(p)} \sum_j (-1)^{j-1} u_{\alpha_1,[a_1]} \ldots u_{\alpha_h,[a_h]}$$
$$\times \varphi_{\beta_1,[b_1]} \ldots \{\varphi_\alpha, \varphi_{\beta_j}\}_{[b_j]} \ldots \varphi_{\beta_{2k},[b_{2k}]} \tag{17b}$$

where p denotes a generic element $([a_1],[a_2],\ldots,[a_h],[b_1],[b_2],\ldots,[b_{2k}])$ of the sets of generalized ordered partitions of $[c]$ and $[d]$, containing fixed numbers (second and third arguments of P) of even and odd subsets, all $[a]$ and $[b]$ being respectively even and odd, except for $[a_j]$ and $[b_j]$ respectively in the first and in the second sum of equation (17b), whose parity is exchanged; partitions are called generalized since several even elements may coincide with the empty set, the corresponding component variables denoting the body of boson supervariables. Finally $\pi(p)=0,1$ is the parity of the permutation of $[c]$ in equation (17a), $[d]$ in equation (17b), corresponding to the given partition, namely of $([a_1],[a_2],\ldots,[a_h],[b_1],[b_2],\ldots,[b_{2k}])$, where naturally ordered subsets of $\{1,2,\ldots,2n\}$ are meant.

If the following is substituted for $M_n \equiv M_{(1,2,\ldots,2n)}$:

$$M_n = \sum_{q\in P(\{1,2,\ldots,2n\},h,2k)} (-1)^{\pi(q)} u_{\alpha_1,[a_1]} u_{\alpha_2,[a_2]} \ldots u_{\alpha_h,[a_h]} \varphi_{\beta_1,[b_1]} \varphi_{\beta_2,[b_2]} \ldots \varphi_{\beta_{2k},[b_{2k}]} \tag{18}$$

the explicit expression of the LHSs of equations (16a) and (16b) is given by

$$\{\psi, M_n\} = \sum_{q\in P(\{1,2,\ldots,2n\},h,2k)} (-1)^{\pi(q)} \Bigg[\sum_j u_{\alpha_1,[a_1]} \ldots \{\psi, u_{\alpha_j,[a_j]}\} \ldots u_{\alpha_h,[a_h]} \varphi_{\beta_1,[b_1]} \ldots \varphi_{\beta_{2k},[b_{2k}]}$$
$$+ \sum_j u_{\alpha_1,[a_1]} \ldots u_{\alpha_h,[a_h]} \varphi_{\beta_1,[b_1]} \ldots \{\psi, \varphi_{\beta_j,[b_j]}\} \ldots \varphi_{\beta_{2k},[b_{2k}]}\Bigg] \tag{19}$$

with ψ replaced by $u_{\alpha,[c]}$ and $\varphi_{\beta,[d]}$ respectively.

Comparing equations (17a) and (17b) with equation (19), it can be proved that compatibility conditions (16a) and (16b), if v, w are either u or φ fields, are fulfilled by

$$\{v_{\gamma,[g]}, w_{\gamma',[g']}\} = 0, [g],[g'] \subset \{1,2,\ldots,2n\} \qquad \text{if } [g]\cup[g'] \neq \{1,2,\ldots,2n\} \tag{20a}$$

$$\{v_{\gamma,[g]}, w_{\gamma',[g']}\} = (-1)^{p_1}(-1)^{p_2}\{v_\gamma, w_{\gamma'}\}_{[g]\cap[g']} \qquad \text{if } [g]\cup[g'] = \{1,2,\ldots,2n\} \tag{20b}$$

where $[g]\cap[g']$ denotes the corresponding naturally ordered subset and $p_1, p_2 = 0,1$ respectively the parity of the permutations

$$(1,2,\ldots,2n) \to ((1,2,\ldots,2n)\backslash[g],[g]) \tag{21a}$$

$$[g'] \to ([g']\backslash[g],[g']\cap[g]). \tag{21b}$$

For details see the appendix, where the proof of antisymmetry and Jacobi identities is also given.

3. Fermion extensions of the KdV equation

The KdV equation is quite popular due to its relevance in the quantization of the Liouville equation, which in turn is relevant to the quantization of the Polyakov string below the traditional critical dimension [15]. The generalization of the known relation between the KdV equation and the Virasoro algebra [16] to the integrable space-supersymmetric KdV equation and the super-Virasoro algebra aroused considerable interest in super-KdV equations [17]. In order to illustrate the proposed setting with some relevant examples, some fermionic versions of this equation are considered in the following.

The proposed fermion extensions of the KdV equation all have the following general form:

$$u_t = 6uu_x - u_{xxx} + e\varphi\varphi_{xx} \tag{22a}$$

$$\varphi_t = f\varphi_{xxx} + gu_x\varphi + hu\varphi_x \tag{22b}$$

where t and x subscripts (which are replaced in the following, where notationally convenient, by dot ($\dot{}$) and prime superscripts ($'$)), respectively denote time and space derivatives, while u and φ are a bosonic and fermionic field respectively, and e, f, g, h are constant parameters (ordinary numbers). In particular the choice

$$e=3 \qquad f=-4 \qquad g=3 \qquad h=6 \tag{23}$$

gives the earliest version, which admits two different super-Hamiltonian realizations [18]. One of them corresponds to the super-Poisson brackets

$$\begin{aligned} &\{u(x), u(y)\}_1 = \delta'(x-y) \qquad \{u(x), \varphi(y)\}_1 = 0 \\ &\{\varphi(x), \varphi(y)\}_1 = a\delta(x-y) \end{aligned} \tag{24}$$

with $a=1$, and the super-Hamiltonian functional

$$H_1[u, \varphi] = \int \mathrm{d}x(u^3 + u_x^2/2 + su\varphi\varphi_x + t\varphi\varphi_{xxx}) \tag{25}$$

with $s=3$, $t=-2$. The alternative Hamiltonian realization is given by the super-Poisson brackets

$$\{u(x), u(y)\}_2 = -\delta'''(x-y) + 2u'(x)\delta(x-y) + 4u(x)\delta'(x-y) \tag{26a}$$

$$\{u(x), \varphi(y)\}_2 = \varphi'(x)\delta(x-y) + 3\varphi(x)\delta'(x-y) \tag{26b}$$

$$\{\varphi(x), \varphi(y)\}_2 = c(-\delta''(x-y) + u(x)\delta(x-y)) \tag{26c}$$

with $c=4$ and, for $b=1$, by the super-Hamiltonian

$$H_2[u, \varphi] = \int \mathrm{d}x(u^2 + b\varphi\varphi_x)/2. \tag{27}$$

The two space-supersymmetric versions of the KdV equation are given by equations (22a) and (22b) with

$$e=-3 \qquad f=-1 \qquad g=3 \qquad h=3 \tag{28a}$$

$$e=-2 \qquad f=-1 \qquad g=2 \qquad h=4 \tag{28b}$$

respectively [17, 19]. The former is super-Hamiltonian with super-Poisson brackets given by equations (26) with $c=-1$ and super-Hamiltonian functional corresponding to H_2 in equation (27) with $b=-1$, while the latter admits a super-Hamiltonian realization with super-Poisson brackets and super-Hamiltonian functional given respectively by equations (24) with $a=-1$ and H_1 in equation (25) with $s=-2$, $t=1/2$.

Once the superfields u and φ are developed according to equations ($9a$) and ($9b$), with the continuous variable x (replacing both discrete indices α and β) omitted, equations ($22a$) and ($22b$) can be rewritten in terms of component fields, as an instance of equations ($11a$)-($11e$), as follows:

$$\dot{u}_B = 6u_B u'_B - u'''_B \tag{29a}$$

$$\dot{\varphi}_{(k)} = f\varphi'''_{(k)} + gu'_B\varphi_{(k)} + hu_B\varphi'_{(k)} \tag{29b}$$

$$\dot{u}_{(h,k)} = 6u_B u'_{(h,k)} + 6u_{(h,k)}u'_B - u'''_{(h,k)} + e(\varphi_{(h)}\varphi''_{(k)} - \varphi_{(k)}\varphi''_{(h)}) \tag{29c}$$

$$\begin{aligned}\dot{\varphi}_{(j,h,k)} = {} & f\varphi'''_{(j,h,k)} + gu'_B\varphi_{(j,h,k)} + g(u'_{(j,h)}\varphi_{(k)} + \mathrm{CP}) + hu_B\varphi'_{(j,h,k)} \\ & + h(u_{(j,h)}\varphi'_{(k)} + \mathrm{CP})\end{aligned} \tag{29d}$$

$$\begin{aligned}\dot{u}_{(i,j,h,k)} = {} & 6(u_B u_{(i,j,h,k)})' + 6(u_{(i,j)}u_{(h,k)} - u_{(i,h)}u_{(j,k)} + u_{(i,k)}u_{(j,h)})' - u'''_{(i,j,h,k)} \\ & + e(\varphi_{(i)}\varphi''_{(j,h,k)} - \varphi_{(j)}\varphi''_{(i,h,k)} + \varphi_{(h)}\varphi''_{(i,j,k)} - \varphi_{(k)}\varphi''_{(i,j,h)}) \\ & - e(\varphi''_{(i)}\varphi_{(j,h,k)} - \varphi''_{(j)}\varphi_{(i,h,k)} + \varphi''_{(h)}\varphi_{(i,j,k)} - \varphi''_{(k)}\varphi_{(i,j,h)}) \\ & \vdots\end{aligned} \tag{29e}$$

According to the general results presented here, closed subsystems of system (29), corresponding to evolution equations for component fields with Grassmann indices ranging in an even subset, are ordinary Hamiltonian systems. To be specific, the ordinary Hamiltonians corresponding to the super-Hamiltonian functionals under consideration are given, according to equation (12), for the case of two Grassmann indices, by

$$\begin{aligned}& H_{1,(1,2)}[u_B, u_{(1,2)}, \varphi_{(1)}, \varphi_{(2)}] \\ & \quad = \int \mathrm{d}x[3u_B^2 u_{(1,2)} + u'_B u'_{(1,2)} + su_B(\varphi_{(1)}\varphi'_{(2)} - \varphi_{(2)}\varphi'_{(1)}) + 2t\varphi_1\varphi'''_2]\end{aligned} \tag{30}$$

$$H_{2,(1,2)}[u_{(B)}, u_{(1,2)}, \varphi_{(1)}, \varphi_{(2)}] = \int \mathrm{d}x[2u_B u_{(1,2)} + b(\varphi_{(1)}\varphi'_{(2)} - \varphi_{(2)}\varphi'_{(1)})]/2. \tag{31}$$

The explicit expression of the Hamiltonian functionals

$$\begin{aligned}H_{j,(1,2,3,4)} = {} & H_{j,(1,2,3,4)}[u_B, u_{(1,2)}, u_{(1,3)}, u_{(1,4)}u_{(2,3)}, u_{(2,4)}, u_{(3,4)}, u_{(1,2,3,4)}, \\ & \varphi_{(1)}, \varphi_{(2)}, \varphi_{(3)}, \varphi_{(4)}, \varphi_{(1,2,3)}, \varphi_{(1,2,4)}, \varphi_{(1,3,4)}, \varphi_{(2,3,4)}] \\ & j = 1, 2\end{aligned} \tag{32}$$

for subsystems given by equations ($29a$)-($29e$) with $i=1, j=2, h=3, k=4$, which can be easily derived from equations (9), (12), (25) and (27), is omitted.

As to the ordinary Poisson brackets corresponding to super-Poisson brackets given in equations (24) and (26), according to the general prescriptions given in equations ($20a$) and ($20b$), for the case of two selected Grassmann indices they read respectively

$$\begin{aligned}& \{u_B(x), u_{(1,2)}(y)\}_1 = \delta'(x-y) \qquad \{\varphi_{(1)}(x), \varphi_{(2)}(y)\}_1 = -a\delta(x-y) \\ & \{u_B, u_B\}_1 = \{u_{(1,2)}, u_{(1,2)}\}_1 = \{\varphi_{(1)}, \varphi_{(1)}\}_1 = \{\varphi_{(2)}, \varphi_{(2)}\}_1 = \{u, \varphi\}_1 = 0\end{aligned} \tag{33}$$

and

$$
\begin{aligned}
&\{u_B, u_B\}_2 = \{u_B, \varphi\}_2 = \{\varphi_{(j)}, \varphi_{(j)}\}_2 = 0 \qquad j = 1, 2 \\
&\{u_B(x), u_{(1,2)}(y)\}_2 = -\delta'''(x-y) + 4u_B(x)\delta'(x-y) + 2u'_B(x)\delta(x-y) \\
&\{u_{(1,2)}(x), u_{(1,2)}(y)\}_2 = 4u_{(1,2)}(x)\delta'(x-y) + 2u'_{(1,2)}(x)\delta(x-y) \\
&\{u_{(1,2)}(x), \varphi_j(y)\}_2 = 3\varphi_j(x)\delta'(x-y) + \varphi'_j(x)\delta(x-y) \qquad j = 1, 2 \\
&\{\varphi_{(1)}(x), \varphi_{(2)}(y)\}_2 = c[\delta''(x-y) - u_B(x)\delta(x-y)].
\end{aligned} \tag{34}
$$

For the case of the four Grassmann indices, Poisson brackets corresponding to equations (24) are given by

$$
\begin{aligned}
&\{\varphi_{(r)}, \varphi_{(s)}\}_1 = \{\varphi_{(r,s,v)}, \varphi_{(w,y,z)}\}_1 = \{\varphi, u\}_1 = 0 \qquad r, s, v, w, y, z = 1, 2, 3, 4 \\
&\{u_B, u_B\}_1 = \{u_{(r,s)}, u_{(r,s)}\}_1 = \{u_{(1,2,3,4)}, u_{(1,2,3,4)}\}_1 = 0 \qquad r, s = 1, 2, 3, 4 \\
&\{\varphi_{(1)}(x), \varphi_{(2,3,4)}(y)\}_1 = -\{\varphi_{(2)}(x), \varphi_{(1,3,4)}(y)\}_1 = \{\varphi_{(3)}(x), \varphi_{(1,2,4)}(y)\}_1 \\
&\qquad\qquad = -\{\varphi_{(4)}(x), \varphi_{(1,2,3)}(y)\}_1 = -\delta(x-y) \\
&\{u_B(x), u_{(1,2,3,4)}(y)\}_1 = \{u_{(1,2)}(x), u_{(3,4)}(y)\}_1 = -\{u_{(1,3)}(x), u_{(2,4)}(y)\}_1 \\
&\qquad\qquad = \{u_{(1,4)}(x), u_{(2,3)}(y)\}_1 = \delta'(x-y)
\end{aligned} \tag{35}
$$

while those corresponding to equations (26) are omitted. It should be remarked that, while Jacobi identities trivially hold for Poisson brackets given by equations (33) and (35), their direct verification for those defined by equations (34) would be a rather tedious job, which can be avoided since, as proved in general, they are a consequence of super-Jacobi identities for the original super-Poisson brackets.

A relevant consequence of the present setting is the possibility of shedding new light on supersymmetry transformations. Consider in fact the two supersymmetric versions of the KdV equation given by equation (22) with parameters chosen according to equations (28*a*) or (28*b*). In order to rewrite the space-supersymmetry (infinitesimal) transformation

$$\delta u(x) = \eta\varphi'(x) \tag{36a}$$

$$\delta\varphi(x) = \eta u(x) \tag{36b}$$

in terms of the component fields, develop the anticommuting parameter η, as an odd supernumber, according to the chosen base of the Grassmann algebra under consideration, namely

$$\eta = \sum_k \frac{1}{(2k-1)!} \eta_{(j_1, j_2 \ldots j_{2k-1})} \zeta_{j_1}\zeta_{j_2} \cdots \zeta_{j_{2k-1}} \tag{37}$$

meaning that it corresponds to a whole (possibly infinite) set of real parameters. The above supersymmetry transformation can then be represented in terms of a multiparameter family of transformations for component fields, as follows:

$$\delta u_B = 0 \tag{38a}$$

$$\delta\varphi_{(i)} = \eta_{(i)} u_B \tag{38b}$$

$$\delta u_{(i,j)} = \eta_{(i)}\varphi'_{(j)} - \eta_{(j)}\varphi'_{(i)} \tag{38c}$$

$$\delta\varphi_{(i,j,h)} = \eta_{(i)}u_{(j,h)} - \eta_{(j)}u_{(i,h)} + \eta_{(h)}u_{(i,j)} + \eta_{(i,j,h)}u_B \tag{38d}$$

$$\delta u_{(i,j,h,k)} = \eta_{(i)}\varphi'_{(j,h,k)} - \eta_{(j)}\varphi'_{(i,h,k)} + \eta_{(h)}\varphi'_{(i,j,k)} - \eta_{(k)}\varphi'_{(i,j,h)} + \eta_{(i,j,h)}\varphi'_{(k)} - \eta_{(i,j,k)}\varphi'_{(h)} + \eta_{(i,h,k)}\varphi'_{(j)} - \eta_{(j,h,k)}\varphi'_{(i)}. \tag{38e}$$

It is easy to check directly that the transformations above are infinitesimal symmetries for equations (29) with parameters chosen either according to equation ($28a$) or ($28b$). To show how supersymmetry transformations can be realized as Hamiltonian symmetries in terms of component fields, consider the closed subsystem for two selected Grassmann indices corresponding to the supersymmetric version defined by equation ($28b$). It is then straightforward to check that the corresponding two-parameter family of transformations obtained from equations ($38a$)-($38c$) for $i, j = 1, 2$ is Hamiltonian with Hamiltonian generators given by

$$\begin{aligned} Q_1[u_B, \varphi_{(1)}, \varphi_{(2)}, u_{(1,2)}] &= \int \mathrm{d}x(\varphi_{(2)}u_B) \\ Q_2[u_B, \varphi_{(1)}, \varphi_{(2)}, u_{(1,2)}] &= \int \mathrm{d}x(-\varphi_{(1)}u_B) \end{aligned} \tag{39}$$

and that invariance now simply reduces to the fact that Q_1 and Q_2 in equation (39) Poisson commute with $H_{1(1,2)}$:

$$\{Q_1, H_{1(1,2)}\}_1 = \{Q_2, H_{1(1,2)}\}_1 = 0. \tag{40}$$

4. Concluding remarks

The most relevant applications of the general result presented here doubtless reside in its quantum counterparts, namely in the corresponding quantization methods for systems with fermions, in terms of component variables. Leaving aside other quantization procedures, path integral quantization of fermion systems (without abandoning the traditional arena of measure-theoretic integration in favour of the merely algebraic integration on Grassmann variables), is the prime candidate to which the present setting can be applied [11]. In this context, however, it is not to be expected that the use of component variables will soon lead to more powerful tools (with respect to the by now traditional ones introduced by Berezin [3]) for analytical calculations, at least in the realm of Hamiltonians, or Lagrangians, quadratic in fermion variables. Monte Carlo simulations, without special tricks to cope with fermion degrees of freedom, should in contrast take substantial advantage from the use of commutative variables only. The present setting in particular gives the possibility of devising a new kind of approximation where a Grassmann algebra with a finite set of generators replaces the original infinite dimensional algebra implied in fermion field theories. This approximation, corresponding to a reduced Fock space with a maximum total occupation number, is under investigation [11].

A further bonus of the proposed setting consists in recasting supersymmetry superalgebras in terms of ordinary (Hamiltonian) symmetry algebras [20], as suggested by the example of the space-supersymmetric KdV equation.

Finally, a by-product of the proposed approach is given by the emergence of whole new classes of ordinary Poisson structures.

It should also be remarked that finite dimensional Grassmann algebras are naturally implied in supersymmetric quantum mechanics [21], where the present setting could be of help in looking for explicit Nicolai maps [22].

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Appendix

Compatibility condition (16a) is obviously fulfilled if equation (20a) holds and if

$$\{u_{\alpha,[a]}, u_{\alpha',[a']}\} = (-1)^{\pi(q)}(-1)^{\pi(p)}\{u_\alpha, u_{\alpha'}\}_{[a]\cap[a']} \qquad [a]\cup[a'] = \{1,2,\ldots,2n\} \tag{A.1a}$$

$$\{u_{\alpha,[a]}, \varphi_{\beta,[b]}\} = (-1)^{\pi(q)}(-1)^{\pi(p)}\{u_\alpha, \varphi_\beta\}_{[a]\cap[b]} \qquad [a]\cup[b] = \{1,2,\ldots,2n\} \tag{A.1b}$$

for every permutation (i.e. partition) q respectively given by

$$(1,2,\ldots,2n) \mapsto ([a_1],[a_2],\ldots,[a_{j-1}],[a'],[a_{j+1}],\ldots,[a_h],[b_1],[b_2],\ldots,[b_{2k}]) \tag{A.2a}$$

$$(1,2,\ldots,2n) \mapsto ([a_1],[a_2],\ldots,[a_h],[b_1],[b_2],\ldots,[b_{j-1}],[b],[b_{j+1}],\ldots,[b_{2k}]) \tag{A.2b}$$

and the permutations p corresponding respectively to

$$[a] \mapsto ([a_1],[a_2],\ldots,[a_{j-1}],[a]\cap[a'],[a_{j+1}],\ldots,[a_h],[b_1],[b_2],\ldots,[b_{2k}]) \tag{A.3a}$$

$$[a] \mapsto ([a_1],[a_2],\ldots,[a_h],[b_1],[b_2],\ldots,[b_{j-1}],[a]\cap[b],[b_{j+1}],\ldots,[b_{2k}]). \tag{A.3b}$$

In order that equation (A.1a) actually defines its LHS, the factor $(-1)^{\pi(q)}(-1)^{\pi(p)}$ has to be independent of the particular partition q, which is actually the case since

$$(-1)^{\pi(q)}(-1)^{\pi(p)} = (-1)^{\pi(p_1)}(-1)^{\pi(p_2)} \tag{A.4}$$

where p_1 and p_2 respectively are the parities of permutations (21a) and (21b) with $[a]$, $[a']$ replacing $[g]$, $[g']$. In fact permutation q can be decomposed as the product of the following permutations:

$$(1,2,\ldots,2n) \mapsto ((1,2,\ldots,2n)\backslash[a],[a]) \tag{A.5}$$

$$\ldots \mapsto ((1,2,\ldots,2n)\backslash[a],[a_1],[a_2],\ldots,[a_{j-1}],[a] \cap[a'],[a_{j+1}],\ldots,[a_h],[b_1],[b_2],\ldots,[b_{2k}]) \tag{A.6}$$

$$\ldots \mapsto ([a_1],[a_2],\ldots,[a_{j-1}],(1,2,\ldots,2n)\backslash[a],[a] \cap[a'],[a_{j+1}],\ldots,[a_h],[b_1],[b_2],\ldots,[b_{2k}]) \tag{A.7}$$

$$\ldots \mapsto ([a_1],[a_2],\ldots,[a_{j-1}],[a'],[a_{j+1}],\ldots,[a_h],[b_1],[b_2],\ldots,[b_{2k}]) \tag{A.8}$$

where dots denote the result of the previous permutation. Moreover permutation (A.5) coincides with (21a) for $[a]$ replacing $[g]$, permutations (A.6) and p in equation (A.3a) have equal parities, permutation (A.7) is always even, and finally the parities of permutations (A.8) and (21b) with $[a]$, $[a']$ replacing $[g]$, $[g']$ coincide. Equation (A.4) then easily follows, which gives equation (20b) for two u variables.

A strictly analogous proof, which is omitted, also works for equation (A.1b) and for what refers to equation (16b). The only variant is encountered in defining the Poisson bracket between two φ fields, in which case the analogue of permutation (A.7) is even or odd according to the parity of the analogue of index j; the ensuing $(-1)^{j-1}$ factor is then compensated by the same factor in equation (17b).

As to antisymmetry, the condition

$$\{u_{\alpha,[a]}, u_{\alpha',[a']}\} = -\{u_{\alpha',[a']}, u_{\alpha,[a]}\} \tag{A.9}$$

is equivalent to $(-1)^{p_1}(-1)^{p_2} = (-1)^{p'_1}(-1)^{p'_2}$ where p_1, p_2 respectively are the parity of permutation (21a) and (21b) with a, a' replacing g, g', and p'_1, p'_2 are obtained from p_1, p_2 by exchanging a and a'; but this is true since, if this equality is rewritten as

$$(-1)^{p_1}(-1)^{p'_2} = (-1)^{p'_1}(-1)^{p_2} \tag{A.10}$$

one easily sees that it is in turn equivalent to the equality of the parity of the following permutation:

$$(1,2,\ldots,2n) \to ((1,2,\ldots,2n)\backslash[a], (1,2,\ldots,2n)\backslash[a'], [a]\cap[a']) \tag{A.11}$$

to the parity of the permutation obtained by exchanging $[a]$ and $[a']$. This last equality obviously holds since both $(1,2,\ldots,2n)\backslash[a]$ and $(1,2,\ldots,2n)\backslash[a']$ are even multi-indices. Also in this case a strictly analogous proof can be given for the antisymmetry relations

$$\{u_{\alpha,[a]}, \varphi_{\beta,[b]}\} = -\{\varphi_{\beta,[b]}, u_{\alpha,[a]}\} \tag{A.12}$$

$$\{\varphi_{\beta,[b]}, \varphi_{\beta',[b']}\} = -\{\varphi_{\beta',[b']}, \varphi_{\beta,[b]}\} \tag{A.13}$$

where the only variant is met in verifying (A.13), which, since super-Poisson brackets are symmetric in fermion fields, i.e.

$$\{\varphi_\beta, \varphi_{\beta'}\} = \{\varphi_{\beta'}, \varphi_\beta\} \tag{A.14}$$

amounts to

$$(-1)^{p_1}(-1)^{p'_2} = -(-1)^{p'_1}(-1)^{p_2} \tag{A.15}$$

finally, equation (A.15) holds due to the oddness of multi-indices $(1,2,\ldots,2n)\backslash[b]$ and $(1,2,\ldots,2n)\backslash[b']$, which appear in the analogue of permutation (A.11).

As to the Jacobi identity, let f, g, h, be three arbitrary, either boson or fermion, supervariables, and, if $[g]$, $[g']$ are Grassmann multi-indices with indices ranging in $\{1,2,\ldots,2n\}$, as in equation (20b), introduce the following notation:

$$\sigma([g],[g']) \equiv p_1 + p_2 \tag{A.16}$$

with p_1, p_2 defined according to equations (20b), (21a) and (21b). Then equation (20b) implies

$$\begin{aligned}\{f_{[\varphi]}, \{g_{[\gamma]}, h_{[\eta]}\}\} + \{h_{[\eta]}, \{f_{[\varphi]}, g_{[\gamma]}\}\} + \{g_{[\gamma]}, \{h_{[\eta]}, f_{[\varphi]}\}\} \\ = [(-1)^{\sigma([\gamma],[\eta])+\sigma([\varphi],[\gamma]\cap[\eta])}\{f, \{g, h\}\} \\ + (-1)^{\sigma([\varphi],[\gamma])+\sigma([\eta],[\varphi]\cap[\gamma])}\{h, \{f, g\}\} \\ + (-1)^{\sigma([\eta],[\varphi])+\sigma([\gamma],[\eta]\cap[\varphi])}\{g, \{h, f\}\}]_{[\varphi]\cap[\gamma]\cap[\eta]}\end{aligned} \tag{A.17}$$

if

$$[\varphi]\cup[\gamma] = [\gamma]\cup[\eta] = [\eta]\cup[\varphi] = (1,2,\ldots,2n). \tag{A.18}$$

(If equation (A.18) is not fulfilled, the corresponding Jacobi identity is trivial since the three terms involved vanish separately.) According to equation (A.16) the exponent in the first term of the RHS of equation (A.17) is given by the sum of the parities of the following permutations:

$$(1,2,\ldots,2n)\to((1,2,\ldots,2n)\backslash[\gamma],[\gamma]) \tag{A.19a}$$

$$[\eta]\to([\eta]\backslash[\gamma],[\eta]\cap[\gamma]) \tag{A.19b}$$

$$(1,2,\ldots,2n)\to((1,2,\ldots,2n)\backslash[\varphi],[\varphi]) \tag{A.19c}$$

$$[\gamma]\cap[\eta]\to(([\gamma]\cap[\eta]\backslash[\varphi],[\varphi]\cap[\gamma]\cap[\eta]) \tag{A.19d}$$

while the sum of the parities of the permutations

$$(1,2,\ldots,2n)\to((1,2,\ldots,2n)\backslash[\varphi],[\varphi]) \tag{A.20a}$$

$$[\gamma]\to([\gamma]\backslash[\varphi],[\gamma]\cap[\varphi]) \tag{A.20b}$$

$$(1,2,\ldots,2n)\to((1,2,\ldots,2n)\backslash[\eta],[\eta]) \tag{A.20c}$$

$$[\varphi]\cap[\gamma]\to(([\varphi]\cap[\gamma])\backslash[\eta],[\varphi]\cap[\gamma]\cap[\eta]) \tag{A.20d}$$

gives the exponent in the second term. Now equality of these two exponents is obviously equivalent to the equality of the parity of the two following permutations:

$$\begin{aligned}(1,2,\ldots,2n)&\to((1,2,\ldots,2n)\backslash[\gamma],[\gamma]\backslash[\varphi],([\gamma]\cap[\varphi])\backslash[\eta],[\varphi]\cap[\gamma]\cap[\eta])\\&=((1,2,\ldots,2n)\backslash[\gamma],(1,2,\ldots,2n)\backslash[\varphi],(1,2,\ldots,2n)\backslash[\eta],\\&\qquad[\varphi]\cap[\gamma]\cap[\eta])\end{aligned} \tag{A.21a}$$

$$\begin{aligned}(1,2,\ldots,2n)&\to((1,2,\ldots,2n)\backslash[\eta],[\eta]\backslash[\gamma],([\eta]\cap[\gamma])\backslash[\varphi],[\varphi]\cap[\gamma]\cap[\eta])\\&=((1,2,\ldots,2n)\backslash[\eta],(1,2,\ldots,2n)\backslash[\gamma],(1,2,\ldots,2n)\backslash[\varphi],\\&\qquad[\varphi]\cap[\gamma]\cap[\eta])\end{aligned} \tag{A.21b}$$

where equation (A.18) was used, which in turn is equivalent to

$$\#([\eta])[\#([\varphi])+\#([\gamma])]=0 \qquad (\mathrm{mod}\ 2) \tag{A.22}$$

where the symbol # is used to denote the number of elements of multi-indices. Equality of exponents in the two other possible couples of terms in the RHS of equation (A.17) obviously leads to two relations obtained from equation (A.22) just by cyclic permutations of Grassmann multi-indices. By using these two relations together with equations (A.22) and (A.17), the generic Jacobi identity is obviously seen to be equivalent to the following:

$$\begin{aligned}&[(-1)^{\#([\varphi])\#([\eta])}\{f,\{g,h\}\}+(-1)^{\#([\eta])\#([\gamma])}\{h,\{f,g\}\}\\&\qquad+(-1)^{\#([\gamma])\#([\varphi])}\{g,\{h,f\}\}]_{[\varphi]\cap[\gamma]\cap[\eta]}=0\end{aligned} \tag{A.23}$$

which is a consequence of graded Jacobi identity in equation (7*a*), since, if $\psi_{[g]}$ is a generic coefficient in the expansion of the supervariable ψ as a power series in Grassmann generators, then obviously

$$\mathfrak{g}(\psi)=\#([g]) \qquad (\mathrm{mod}\ 2). \tag{A.24}$$

References

[1] Schwinger J 1953 *Phil. Mag.* **44** 1171
Martin I L 1959 *Proc. Roy. Soc.* A **251** 536

[2] Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill)
Schulman L S, 1981 *Techniques and Applications of Path Integration*, (New York: Wiley) and references quoted therein

[3] Berezin F A 1966 *The Method of Second Quantization* (New York: Academic)

[4] Casalbuoni R 1976 *Nuovo Cimento* A **33** 389
Berezin F A and Marinov M S 1977 *Ann. Phys.* **104** 336

[5] Ramond P 1971 *Phys. Rev.* D **13** 2415
Neveu A and Schwarz J H 1971 *Nucl. Phys.* B **31** 86
Green M B and Schwarz J H 1982 *Phys. Lett.* **109B** 444

[6] De Filippo S, Landi G, Marmo G and Vilasi G 1989 *Phys. Lett.* **220B** 576 and references quoted therein

[7] De Filippo S, Landi G, Marmo G and Vilasi G 1989 *Ann. Inst. H Poincaré* **50** 205 and references quoted therein

[8] Feynman R P 1948 *Rev. Mod. Phys.* **20** 367

[9] Symanzik K 1966 *J. Math. Phys.* **7** 510
Glimm J and Jaffe A 1981 *Quantum Physics—A Functional Integral Point of View* (New York: Springer) and references quoted therein

[10] De Filippo S 1991 *Phys. Lett.* **265B** 127

[11] De Filippo S work in progress

[12] Creutz M 1983 *Quarks, Gluons and Lattices* (Cambridge: Cambridge University Press)
Schmidt K E and Kalos M H 1987 *Applications of Monte Carlo Method in Statistical Physics* ed K Binder (Berlin: Springer). See also chapter 10 in the same volume

[13] DeWitt B 1984 *Supermanifolds* (Cambridge: Cambridge University Press)
Berezin F A 1987 *Introduction to Superanalysis* (Dordrecht: Reidel)

[14] De Filippo S in preparation

[15] Gervais J L 1985 *Phys. Lett.* **160B** 277 and references quoted therein

[16] Gervais J L and Neveu A 1982 *Nucl. Phys.* B **209** 125

[17] Manin Yu I and Radul A O 1985 *Commun. Math. Phys.* **98** 65
Chaichian M and Kulish P P 1987 *Phys. Lett.* **183B** 169
Mathieu P 1988 *Phys. Lett.* **203B** 287

[18] Kupershmidt B A 1984 *Phys. Lett.* **102A** 213

[19] Mathieu P 1988 *J. Math. Phys.* **29** 2499

[20] De Filippo S in preparation

[21] Witten E 1981 *Nucl. Phys.* B **185** 513
Graham R and Roekaerts D 1986 *Phys. Rev.* D **34** 2312 and references therein

[22] Nicolai H 1980 *Phys. Lett.* **89B** 341; *Nucl. Phys.* B **176** 419